

# On Induced Colourful Paths in Triangle-free Graphs

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## Abstract

Given a graph  $G = (V, E)$  whose vertices have been properly coloured, we say that a path in  $G$  is *colourful* if no two vertices in the path have the same colour. It is a corollary of the Gallai-Roy Theorem that every properly coloured graph contains a colourful path on  $\chi(G)$  vertices. It is interesting to think of what analogous result one could obtain if one considers induced colourful paths instead of just colourful paths. We explore a conjecture that states that every properly coloured triangle-free graph  $G$  contains an induced colourful path on  $\chi(G)$  vertices. As proving this conjecture in its fullest generality seems to be difficult, we study a special case of the conjecture. We show that the conjecture is true when the girth of  $G$  is equal to  $\chi(G)$ . Even this special case of the conjecture does not seem to have an easy proof: our method involves a detailed analysis of a special kind of greedy colouring algorithm. This result settles the conjecture for every properly coloured triangle-free graph  $G$  with girth at least  $\chi(G)$ .

**Keywords:** Induced Path, Colourful Path, Triangle-free Graph

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## 1. Introduction

All graphs considered in this paper are simple, undirected and finite. For a graph  $G = (V, E)$ , we denote the vertex set of  $G$  by  $V(G)$  and the edge set of  $G$  by  $E(G)$ . A function  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  is said to be a *proper  $k$ -colouring* of  $G$  if for any edge  $uv \in E(G)$ , we have  $c(u) \neq c(v)$ . A graph is *properly coloured*, if it has an associated proper  $k$ -colouring  $c$  specified (for some  $k$ ). The minimum integer  $k$  for which a graph  $G$  has a proper  $k$ -colouring is the *chromatic number* of  $G$ , denoted by  $\chi(G)$ . A subgraph  $H$  of a properly coloured graph  $G$  is said to be *colourful* if no two vertices of  $H$  have the same colour. If a colourful subgraph  $H$  of  $G$  is also an induced subgraph, then we say that  $H$  is an *induced colourful subgraph* of  $G$ .

It is a corollary of the classic Gallai-Roy Theorem [3] that every (not necessarily optimally) properly coloured graph  $G$  has a colourful path on  $\chi(G)$  vertices (an alternative proof for this is given in Theorem 4). We are interested in the question of when one can find colourful paths that are also induced in a given properly coloured graph. Note that the colourful path on  $\chi(G)$  vertices that should exist in any properly coloured graph  $G$  may not always be an induced path. In fact, when  $G$  is a complete graph, there is no induced path on more than two vertices in the graph. The following hitherto unpublished conjecture is due to N. R. Aravind.

**Conjecture 1** ([2]). Let  $G$  be a triangle-free graph that is properly coloured. Then there is an induced colourful path on  $\chi(G)$  vertices in  $G$ .

Surprisingly, despite being known to many researchers for well over two years, the conjecture has remained open even for the special case when  $\chi(G) = 4$ . Note that Conjecture 1 is readily seen to be true for any triangle-free graph  $G$  with  $\chi(G) = 3$ , because the colourful path guaranteed to exist in  $G$  by the Gallai-Roy Theorem is also an induced path in  $G$ . In this paper, we first prove Conjecture 1 for the case when  $\chi(G) = 4$ . We then extend this proof to show that the conjecture holds for any triangle-free graph  $G$  with  $g(G) \geq \chi(G)$ , where  $g(G)$  is the *girth* of  $G$ , or the length of the smallest cycle in  $G$ .

A necessary condition for Conjecture 1 to hold is the presence of an induced path on  $\chi(G)$  vertices in any triangle free graph  $G$ . Indeed something stronger is known to be true: each vertex in a triangle-free graph  $G$  is the starting point of an induced path on  $\chi(G)$  vertices [4]. There have been several investigations on variants of the Gallai-Roy Theorem [1, 7]. Every connected graph  $G$  other than  $C_7$  admits a proper  $\chi(G)$ -colouring such that every vertex of  $G$  is the beginning of a (not necessarily induced) colourful path on  $\chi(G) - 1$  vertices [1]. Concerning induced trees, Gyárfás conjectured that there exists an integer-valued function  $f$  defined on the finite trees with the property that

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every triangle-free graph  $G$  with  $\chi(G) = f(T)$  contains  $T$  as an induced subgraph. This was proven true for trees of radius two by Gyárfás, Szemerédi, and Tuza [5]. A stronger version of the Gallai-Roy Theorem that guarantees an induced directed path on  $\chi(G)$  vertices in any directed graph  $G$  would have easily implied Conjecture 1. Clearly, such a theorem cannot be true for every directed graph. But Kierstead and Trotter [6] show that no such result can be obtained even if the underlying undirected graph of  $G$  is triangle-free. They show that for every natural number  $k$ , there exists a digraph  $G$  such that its underlying undirected graph is triangle-free and has chromatic number  $k$ , but  $G$  has no induced directed path on 4 vertices.

## 2. Preliminaries

Notation used in this paper is the standard notation used in graph theory (see e.g. [3]). We shall now describe a special greedy colouring procedure for an already coloured graph that will later help us in proving our main result.

**The refined greedy algorithm.** Given a properly coloured graph  $G$  with the colouring  $\beta$ , we will construct a new proper colouring  $\alpha : V(G) \rightarrow \mathbb{N}^{>0}$  of  $G$ , using the algorithm given below. Let  $b_1 < b_2 < \dots < b_t$  be the colours used by  $\beta$ .

For every vertex  $v \in V(G)$ , set  $\alpha(v) \leftarrow 0$

**for**  $i$  from 1 to  $t$  **do**

**for** vertex  $v$  with  $\beta(v) = b_i$  and  $\alpha(v) = 0$  **do**

        Colour  $v$  with the least positive integer not present in its neighbourhood, i.e., set  $\alpha(v) \leftarrow \min(\mathbb{N}^{>0} \setminus \{\alpha(u) : u \in N(v)\})$ .

**Definition 2** (Decreasing path). A path  $u_1 u_2 \dots u_l$  in  $G$  is said to be a “decreasing path” if for  $2 \leq i \leq l$ ,  $\alpha(u_i) < \alpha(u_{i-1})$  and  $\beta(u_i) < \beta(u_{i-1})$ .

**Lemma 3.** Let  $v \in V(G)$  and  $X \subseteq \{1, 2, \dots, \alpha(v) - 1\}$ . Then there is a decreasing path  $vu_{|X|}u_{|X|-1} \dots u_1$  in  $G$  such that for  $1 \leq i \leq |X|$ ,  $\alpha(u_i) \in X$ .

This lemma directly shows that there is a colourful path on  $\chi(G)$  vertices in every properly coloured graph  $G$  (without using the Gallai-Roy Theorem).

**Theorem 4.** If  $G$  is any graph whose vertices are properly coloured, then there is a colourful path on  $\chi(G)$  vertices in  $G$ .

*Proof.* Let  $\beta$  denote the proper colouring of  $G$ . Run the refined greedy algorithm on  $G$  to generate the colouring  $\alpha$ . Clearly, the algorithm will use at least  $\chi(G)$  colours as the colouring  $\alpha$  generated by the algorithm is also a proper colouring of  $G$ . Let  $v$  be any vertex in  $G$  with  $\alpha(v) = \chi(G)$ . Now consider the set  $X = \{1, 2, \dots, \chi(G) - 1\}$ . By Lemma 3, there is a path on  $\chi(G)$  vertices starting at  $v$  on which the colours in the colouring  $\beta$  are strictly decreasing. This path is a colourful path on  $\chi(G)$  vertices in  $G$ .  $\square$

**Corollary 5.** Any properly coloured graph  $G$  with  $g(G) > \chi(G)$  has an induced colourful path on  $\chi(G)$  vertices.

*Proof.* If  $g(G) > \chi(G)$ , then the colourful path given by Theorem 4 is an induced path in  $G$ .  $\square$

This implies that the conjecture is true for all triangle-free graphs with chromatic number at most 3. It also implies that in order to prove Conjecture 1, one only has to consider graphs  $G$  with  $g(G) \leq \chi(G)$ . The main result of this paper is that Conjecture 1 holds true for all triangle-free graphs  $G$  with  $g(G) = \chi(G)$ .

## 3. Induced colourful paths in graphs with girth equal to chromatic number

In this section, we shall prove our main result, given by the theorem below.

**Theorem 6.** Let  $G$  be a graph with  $g(G) = \chi(G) = k$ , where  $k \geq 4$ , and whose vertices have been properly coloured. Then there exists an induced colourful path on  $k$  vertices in  $G$ .

Note that we can assume that  $G$  is connected, because if the theorem holds for connected graphs, then it will hold for the connected component of  $G$  with chromatic number equal to  $k$  and hence also for  $G$ . Let  $\beta : V(G) \rightarrow \{1, 2, \dots, t\}$  denote the proper colouring of  $G$  that is given.

A  $k$ -cycle in  $G$  in which no colour repeats is said to be a *colourful  $k$ -cycle*, sometimes shortened to just “colourful cycle”. Notice that every colourful cycle in  $G$  is also an induced cycle as  $g(G) = k$ . From here onwards, we shorten “colourful path on  $k$  vertices” to just “colourful path”.

Suppose that there is no induced colourful path on  $k$  vertices in  $G$ .

**Observation 7.** *Since  $g(G) = k$ , if  $y_1 y_2 \dots y_k$  is a colourful path on  $k$  vertices in  $G$ , then the edge  $y_1 y_k \in E(G)$ . Thus,  $y_1 y_2 \dots y_k y_1$  is a colourful  $k$ -cycle in  $G$ .*

Let  $\alpha$  be a proper colouring of  $G$  generated by running the refined greedy algorithm on  $G$ . We shall refer to the colours of the colouring  $\alpha$  as “labels”. From here onwards, we shall reserve the word “colour” to refer to a colour in the colouring  $\beta$ . As before, whenever we say that a path or a cycle is “colourful”, we are actually saying that it is colourful in the colouring  $\beta$ .

We say that a path with no repeating colours is an “almost decreasing path” if the subpath induced by the vertices other than the starting vertex is a decreasing path. Note that any decreasing path is also an almost decreasing path.

The proof of Theorem 6 is split into two cases: when  $k = 4$  and when  $k > 4$ .

### 3.1. Case when $k = 4$

In this case, we have  $\chi(G) = g(G) = 4$ .

As  $\alpha$  is also a proper colouring of  $G$ , we know that there exists a vertex  $v$  in  $G$  with label 4. By Lemma 3, there exists a decreasing path  $v_4 v_3 v_2 v_1$  where  $v_4 = v$  and for  $1 \leq i \leq 3$ , we have  $\beta(v_i) < \beta(v_{i+1})$  and  $\alpha(v_i) = i$ . Again by Lemma 3, we have a path  $vv'_2 v'_1$  in which we have  $\beta(v'_1) < \beta(v'_2) < \beta(v)$ ,  $\alpha(v'_2) = 2$  and  $\alpha(v'_1) = 1$ . Note that  $v'_2 \neq v_2$  and  $v'_1 \neq v_1$  (as otherwise  $vv'_2 v_1 v$  would be a triangle in  $G$ ). This means that the vertices in  $\{v_4, v_3, v_2, v_1, v'_2, v'_1\}$  are all pairwise distinct. Let  $\beta(v_i) = b_i$  for each  $i$ , where  $1 \leq i \leq 4$ . We shall call the colours  $b_1, b_2, b_3, b_4$  “primary colours”. Clearly, as  $v_4 v_3 v_2 v_1$  is a decreasing and hence colourful path, by Observation 7, we have  $v_1 v_4 \in E(G)$ .

**Claim 8.**  $\beta(v'_2) = b_2$  and  $\beta(v'_1) = b_1$ .

*Proof.* Suppose that  $\beta(v'_2) \neq b_2$ . Then we have that either the path  $v'_2 v_4 v_3 v_2$  or the path  $v'_2 v_4 v_1 v_2$  is colourful, which implies that  $v'_2 v_2 \in E(G)$ , a contradiction since  $\alpha(v'_2) = \alpha(v_2)$ . Therefore we have  $\beta(v'_2) = b_2$ . Similarly if  $\beta(v'_1) \neq b_1$ , then the path  $v'_1 v'_2 v_4 v_1$  is colourful, which implies that  $v'_1 v_1 \in E(G)$ , a contradiction since  $\alpha(v'_1) = \alpha(v_1)$ . Thus we have  $\beta(v'_1) = b_1$ .  $\square$

Now notice that the path  $v'_1 v'_2 v_4 v_3$  is colourful and hence we have that  $v'_1 v_3 \in E(G)$ . We call the vertices in the set  $\{v_4, v_3, v_2, v_1, v'_2, v'_1\}$  “forced vertices”. Any other vertex in the graph will be called an “optional vertex”. The following observations about forced vertices are easy to verify.

**Observation 9.** *For any forced vertex  $w$ , we have  $\beta(w) = b_{\alpha(w)}$ .*

**Observation 10.** *Every forced vertex is in a colourful cycle containing only forced vertices and which also contains the vertices  $v_3$  and  $v_4$ .*

**Observation 11.** *For any forced vertex  $w$  and set of primary colours  $X \subseteq \{b_i \mid b_i < \beta(w)\}$ , there exists a decreasing path starting from  $w$  having only forced vertices and whose vertices other than  $w$  have exactly the colours in  $X$ .*

**Observation 12.** *For  $w \in \{v_3, v_4\}$  and for any set  $X$  of primary colours, there exists an almost decreasing path containing only forced vertices and whose vertices other than  $w$  see exactly the colours in  $X \setminus \{\beta(w)\}$ .*

For a vertex  $w$ , we define  $\rho(w) = v_3$  if  $\beta(w) = b_4$  and  $\rho(w) = v_4$  otherwise.

**Lemma 13.** *If an optional vertex is adjacent to a forced vertex, then it is adjacent to at least two forced vertices. Moreover, there is a colourful cycle containing the optional vertex in which every other vertex is a forced vertex.*

*Proof.* Let  $w$  be an optional vertex that is adjacent to a forced vertex  $u$ . From Observation 10, there is a colourful cycle  $C$  that contains  $u$  and  $\rho(w)$ . Let  $P$  be a subpath of  $C$  with endvertices  $\rho(w)$  and  $u$  which does not contain the colour  $\beta(w)$  (note that if  $\rho(w) = u$ , then  $P$  consists of just the single vertex  $u = \rho(w)$ ). From Observation 12, there exists an almost decreasing path  $P'$  from  $\rho(w)$  whose vertices other than  $\rho(w)$  see  $4 - |V(wu \cup P)|$  primary colours not present in the path  $wu \cup P$ . Let  $x$  be the endpoint of this path (note that  $P'$  can be the single vertex  $x = \rho(w)$  in case the path  $wu \cup P$  already contains 4 vertices). Clearly,  $wu \cup P \cup P'$  is a colourful path and hence  $wx \in E(G)$ . As it can be easily seen that  $x$  is a forced vertex that is different from  $u$ , we now have at least two forced vertices in  $N(w)$ . Also,  $wu \cup P \cup P' \cup xw$  is a colourful cycle with the required properties.  $\square$

**Lemma 14.** *All optional vertices are adjacent to a forced vertex.*

*Proof.* Consider the set of all optional vertices that have no forced vertices as neighbours. Let  $w$  be a vertex in this set that is closest to a forced vertex. As  $G$  is connected,  $w$  has a neighbour  $w'$  such that  $N(w')$  contains a forced vertex. From Lemma 13, there is a colourful cycle  $C$  containing  $w'$  in which all other vertices are forced vertices. Let  $z$  be the vertex in  $C$  which has highest colour in  $C$  other than  $\beta(w)$  and  $\beta(w')$ . From the observation in the previous sentence, we know that  $z$  is a forced vertex. Let  $P$  be a subpath of  $C$  with endvertices  $w'$  and  $z$  that does not contain  $\beta(w)$ . By Observation 11, from  $z$ , there is a decreasing path  $P'$  whose vertices other than  $z$  see  $4 - |V(ww' \cup P)|$  primary colours not seen on the path  $ww' \cup P$ . Let  $x$  be the endpoint of  $P'$ . As  $P'$  is a decreasing path starting from the forced vertex  $z$ , we have that  $x$  is a forced vertex. Now,  $ww' \cup P \cup P'$  is a colourful path and hence  $wx \in E(G)$ . But now  $x$  is a forced vertex in  $N(w)$ , contradicting the assumption that  $w$  had no forced vertices in its neighbourhood.  $\square$

Let  $S_1$  denote the set of optional vertices adjacent to at least one of the forced vertices  $\{v_4, v_2, v'_1\}$  and let  $S_2$  denote the set of optional vertices adjacent to at least one of the forced vertices  $\{v_3, v_1, v'_2\}$ .

**Lemma 15.** (i)  $S_1$  and  $S_2$  are disjoint, and

(ii)  $S_1$  and  $S_2$  are both independent sets.

*Proof.* First let us show that  $S_1$  and  $S_2$  are disjoint. Suppose that there is a vertex  $w \in S_1 \cap S_2$ . We know that there are two forced vertices  $x$  and  $y$  in  $N(w)$  such that  $x \in \{v_4, v_2, v'_1\}$  and  $y \in \{v_3, v_1, v'_2\}$ . As  $G$  is triangle-free, we only have the two possibilities  $(x = v'_1, y = v_1)$  or  $(x = v_2, y = v'_2)$ . Suppose that  $x = v'_1$  and  $y = v_1$ . If  $\beta(w) \neq b_4$ , then as at least one of the paths  $ww'_1v'_2v_4$  or  $ww'_1v_3v_4$  is a colourful path, we have  $ww_4 \in E(G)$ . But this is a contradiction as  $ww_4v_1$  is a triangle in  $G$ . Therefore, we can conclude that  $\beta(w) = b_4$ . But now, the path  $ww_1v_2v_3$  is colourful, implying that  $ww_3 \in E(G)$ . This is a contradiction as  $ww_3v'_1$  is now a triangle in  $G$ . The other case when  $x = v_2$  and  $y = v'_2$  is symmetric. This proves (i).

This tells us that for each vertex  $w \in S_1$ , the forced vertices in  $N(w)$  all lie in  $\{v_4, v_2, v'_1\}$  and for each vertex  $w' \in S_2$ , the forced vertices in  $N(w')$  all lie in  $\{v_3, v_1, v'_2\}$ . Since we know from Lemma 13 and Lemma 14 that each vertex in  $S_1 \cup S_2$  has at least two forced vertices in their neighbourhood, we can conclude that each vertex in  $S_1$  has at least two neighbours from  $\{v_4, v_2, v'_1\}$  and that each vertex in  $S_2$  has at least two neighbours from  $\{v_3, v_1, v'_2\}$ . This means that for any two  $w, w' \in S_1$ , there is at least one vertex in  $\{v_4, v_2, v'_1\}$  that is a neighbour of both  $w$  and  $w'$ . As  $G$  is triangle-free, we can conclude that  $ww' \notin E(G)$ . For the same reason, for any two vertices  $w, w' \in S_2$ , we have  $ww' \notin E(G)$ . This proves (ii).  $\square$

From Lemma 15(i), we know that there are no edges between  $S_1$  and  $\{v_3, v_1, v'_2\}$ . Similarly, there are no edges between  $S_2$  and  $\{v_4, v_2, v'_1\}$ . Now, by Lemma 15(ii), we have that  $S_1 \cup \{v_3, v_1, v'_2\}$  is an independent set and  $S_2 \cup \{v_4, v_2, v'_1\}$  is an independent set. Since from Lemma 14, we know that  $V(G) = S_1 \cup S_2 \cup \{v_4, v_3, v_2, v_1, v'_2, v'_1\}$ , this tells us that  $G$  is bipartite, which contradicts the assumption that  $\chi(G) = 4$ . Therefore, there can be no properly coloured graph  $G$  such that  $g(G) = \chi(G) = 4$  with no induced colourful path on 4 vertices. This completes the proof of Theorem 6 for the case  $k = 4$ .

### 3.2. Case when $k > 4$

The proof for this case also follows the same general pattern as the case  $k = 4$ , but more technicalities are involved.

**Lemma 16.** *Let  $y_1y_2 \dots y_ky_1$  be a colourful  $k$ -cycle. Let  $z \in N(y_i) \setminus \{y_{i-1}, y_{i+1}\}$  for some  $i \in \{1, 2, \dots, k\}$ . Then  $\beta(z) \in \{\beta(y_1), \dots, \beta(y_k)\} \setminus \{\beta(y_{i-1}), \beta(y_i), \beta(y_{i+1})\}$ . (Here we assume that  $y_{i+1} = y_1$  when  $i = k$  and that  $y_{i-1} = y_k$  when  $i = 1$ .)*

*Proof.* Clearly,  $z \notin \{y_1, y_2, \dots, y_k\}$  as every colourful cycle is an induced cycle. Suppose  $\beta(z) \notin \{\beta(y_1), \dots, \beta(y_k)\} \setminus \{\beta(y_{i-1}), \beta(y_i), \beta(y_{i+1})\}$ . Clearly,  $\beta(z) \neq \beta(y_i)$ . Suppose that  $\beta(z) \neq \beta(y_{i+1})$ . Then observe that  $zy_i y_{i+1} \dots y_k y_1 \dots y_{i-2}$  is a colourful path on  $k$  vertices and hence  $zy_{i-2} \in E(G)$ . This implies that  $zy_i y_{i-1} y_{i-2} z$  is a 4-cycle in  $G$ , which is a contradiction. If  $\beta(z) = \beta(y_{i+1})$ , then we have  $\beta(z) \neq \beta(y_{i-1})$ . In this case, the path  $zy_i y_{i-1} \dots y_1 y_k \dots y_{i+2}$  is a colourful path and the same reasoning as above tells us that there is a 4-cycle  $zy_i y_{i+1} y_{i+2} z$  in  $G$ , which is again a contradiction.  $\square$

**Corollary 17.** *Let  $y_1 y_2 \dots y_k y_1$  be a colourful  $k$ -cycle. Let  $z \in N(y_i)$  for some  $i \in \{1, 2, \dots, k\}$ . Then  $\beta(z) \in \{\beta(y_1), \dots, \beta(y_k)\}$ .*

**The vertex  $v$ :** Fix  $v$  to be a vertex which has the largest label. Since  $\alpha$  is also a proper vertex colouring of  $G$ , it should use at least  $k$  labels. In other words,  $\alpha(v) \geq k$ .

**Primary cycle:** By applying Lemma 3 to  $v$  and the set of labels  $\{1, 2, \dots, k-1\}$ , we can conclude that there exists a decreasing path  $v_k v_{k-1} \dots v_1$  where  $v_k = v$  and such that  $\alpha(v_i) = i$  for all  $i < k$  and  $\beta(v_i) < \beta(v_{i+1})$  for all  $1 \leq i \leq k-1$ . Since this path is colourful, by Observation 7,  $v v_{k-1} v_{k-2} \dots v_1 v$  is a colourful cycle, which we shall call the “primary cycle”. For  $1 \leq i \leq k$ , we shall denote by  $b_i$  the colour  $\beta(v_i)$ . The set of colours  $\{b_k, b_{k-1}, \dots, b_1\}$  shall be called the set of “primary colours”.

**Lemma 18.**  $\alpha(v) = k$ . Hence, for all  $i$ ,  $\alpha(v_i) = i$ .

*Proof.* Suppose for the sake of contradiction that  $\alpha(v) > k$ . By Lemma 3, there exists a decreasing path  $y_{k+1} y_k \dots y_1$  where  $y_{k+1} = v$  and for  $1 \leq i \leq k$ , we have  $\alpha(y_i) = i$  and  $\beta(y_i) < \beta(y_{i+1})$ . As the paths  $y_{k+1} y_k \dots y_2$  and  $y_k y_{k-1} \dots y_1$  are both colourful, it must be the case that  $y_{k+1} y_2, y_k y_1 \in E(G)$ . But then,  $y_{k+1} y_2 y_1 y_k y_{k+1}$  is a cycle on four vertices in  $G$ , which is a contradiction.  $\square$

**Forced and optional vertices:** A vertex  $u \in V(G)$  is said to be a “forced vertex” if there is a decreasing path from  $v$  to  $u$ . Any vertex of  $G$  that is not forced is said to be an “optional vertex”.

**Lemma 19.** *For each  $i \in \{1, 2, \dots, k-1\}$  there is exactly one vertex  $u_i$  in  $N(v)$  with label  $i$ . Moreover,  $\beta(u_i) = b_i$  and there is a colourful cycle  $C_i$  containing  $u_i$  and  $v$  that contains only forced vertices.*

*Proof.* Observe that because the refined greedy algorithm assigned  $\alpha(v)$  to be  $k$ , it must be the case that for every  $i \in \{1, 2, \dots, k-1\}$ , there exists a vertex, which we shall call  $u_i$ , in  $N(v)$  such that  $\alpha(u_i) = i$  and  $\beta(u_i) < b_k$ . We shall choose  $u_{k-1}$  to be  $v_{k-1}$ . Because  $u_i$  is adjacent to  $v$  which is on the primary cycle, by Corollary 17, we know that  $\beta(u_i)$  is a primary colour.

We claim that  $\beta(u_i) = b_i$  and that there is a colourful cycle containing  $v$  and  $u_i$  that contains only forced vertices. We shall use backward induction on  $i$  to prove this. Consider the base case when  $i = k-1$ . Since  $u_{k-1} = v_{k-1}$ , we know that  $\beta(u_{k-1}) = b_{k-1}$  and that there is a colourful cycle (the primary cycle) that contains  $u_{k-1}$  and  $v$  and also contains only forced vertices. Thus the claim is true for the base case. Let us assume that the claim has been proved for  $u_{k-1}, u_{k-2}, \dots, u_{i+1}$ . If  $\beta(u_i) = b_j > b_i$ , then  $b_j \in \{b_{i+1}, b_{i+2}, \dots, b_{k-1}\}$ . By the induction hypothesis, we know that the vertex  $u_j \in N(v)$  has  $\beta(u_j) = b_j$  and that there is a colourful cycle  $C_j$  containing  $u_j$  and  $v$ . Note that  $u_j \neq u_i$  (as  $\alpha(u_i) \neq \alpha(u_j)$ ), but  $\beta(u_j) = \beta(u_i) = b_j$ . Therefore, as  $C_j$  contains  $u_j$  and is a colourful cycle, it cannot contain  $u_i$ . Since  $u_i$  is adjacent to  $v$  which is on  $C_j$ , and  $\beta(u_i) = b_j$ , we now have a contradiction to Lemma 16 (note that  $u_j v$  is an edge of  $C_j$  as every colourful cycle is a chordless cycle). So it has to be the case that  $\beta(u_i) \leq b_i$ . By Lemma 3, there exists a path  $y_i y_{i-1} y_{i-2} \dots y_1$ , where  $y_i = u_i$ , such that for  $1 \leq j \leq i-1$ ,  $\alpha(y_j) = j$  and  $\beta(y_j) < \beta(y_{j+1})$ . Notice that  $y_1 y_2 \dots y_i v_k v_{k-1} \dots v_{i+1}$  is a colourful path and therefore by Observation 7,  $C_i = y_1 y_2 \dots y_i v_k v_{k-1} \dots v_{i+1} y_1$  is a colourful cycle containing both  $u_i$  and  $v$ . Since  $v_i$  is adjacent to  $v_{i+1}$  which is on  $C_i$ , by Corollary 17, we know that there is some vertex  $z$  on  $C_i$  such that  $\beta(z) = b_i$ . Clearly,  $z \in \{y_i, y_{i-1}, \dots, y_1\}$ . If  $z \in \{y_{i-1}, \dots, y_1\}$ , then  $\beta(y_i) > \beta(z) = b_i$ , which is a contradiction to our earlier observation that  $\beta(u_i) \leq b_i$  (recall that  $u_i = y_i$ ). Therefore,  $z = y_i$ , which implies that  $\beta(u_i) = b_i$ . Notice that each  $y_j \in \{y_i, y_{i-1}, \dots, y_1\}$ , because of the decreasing path  $v y_i y_{i-1} \dots y_j$ , is a forced vertex. Thus,  $C_i$  is a colourful cycle containing  $u_i$  and  $v$  that contains only forced vertices. This shows that for any  $i \in \{1, 2, \dots, k-1\}$ ,  $\beta(u_i) = b_i$  and there is a colourful cycle containing  $v$  and  $u_i$  that contains only forced vertices.

We shall now show that  $u_i$  is the only vertex in  $N(v)$  which has the label  $i$ . Suppose that there is a vertex  $u \in N(v)$  such that  $\alpha(u) = i$  and  $u \neq u_i$ . Since  $u$  is adjacent to a colourful cycle containing only primary colours (the



primary cycle), we can conclude from Corollary 17 that  $\beta(u)$  is a primary colour. Therefore,  $\beta(u) = b_j$  for some  $j \in \{1, 2, \dots, k-1\}$ . From what we observed above,  $\beta(u_j) = b_j$  and there exists a colourful cycle  $C_j$  containing the vertices  $v$  and  $u_j$ . Note that  $u_j \neq u$  since if  $j \neq i$ , then  $u_j$  and  $u$  have different labels and if  $j = i$ , we know that  $u_j \neq u$  (as we have assumed that  $u_i \neq u$ ). Hence  $u$  is not in  $C_j$  (as  $C_j$  already has a vertex  $u_j$  with  $\beta(u_j) = b_j$ ) but is adjacent to it. But now  $C_j$  and  $u$  contradict Lemma 16 as  $u_j v$  is an edge of  $C_j$ . Therefore,  $u$  cannot exist.  $\square$

**Corollary 20.** *Let  $C$  be any colourful cycle containing  $v$ . Then  $C$  contains only primary colours.*

*Proof.* Notice that from Lemma 19, we know that for every primary colour  $b_j \in \{b_1, b_2, \dots, b_{k-1}\}$ , there is a vertex  $u_j$  with  $\beta(u_j) = b_j$  that is adjacent to  $v$ . Because  $v$  is in  $C$ , we can apply Corollary 17 to  $C$  and  $u_j$  to conclude that  $b_j$  is present in  $C$ . This means that every primary colour appears on at least one vertex of  $C$ . Since  $C$  was a  $k$ -cycle, this means that  $C$  contains only primary colours.  $\square$

**Lemma 21.** *If  $u \in V(G)$  is a forced vertex such that  $\alpha(u) = i$ , then  $\beta(u) = b_i$ . Moreover, if  $P$  is any decreasing path from  $v$  to  $u$ , then there is a colourful cycle which has  $P$  as a subpath and contains only forced vertices and primary colours.*

*Proof.* Consider a forced vertex  $u$ . We shall prove the statement of the lemma for  $u$  by backward induction on  $\alpha(u)$ . The statement is true for  $\alpha(u) \in \{k, k-1\}$  as there is only one forced vertex each with labels  $k$  and  $k-1$ —which are  $v$  and  $v_{k-1}$  respectively (recall that from Lemma 19,  $u_{k-1} = v_{k-1}$  is the only vertex in  $N(v)$  with label  $k-1$ ). Also, note that they are both in a colourful cycle (the primary cycle) that satisfies the required conditions. Let us assume that the statement of the lemma has been proved for  $\alpha(u) \in \{k, k-1, \dots, i+1\}$ . Let us look at the case when  $\alpha(u) = i$ . Let  $z$  be the predecessor of  $u$  in the path  $P$  and let  $P_z$  be the subpath of  $P$  that starts at  $v$  and ends at  $z$ . Let  $\alpha(z) = j$ . By the induction hypothesis,  $\beta(z) = b_j$  and  $z$  is in a colourful cycle  $C$  that contains only primary colours. By Corollary 17, we can infer that  $\beta(u)$  is a primary colour. Since  $P$  was a decreasing path,  $\beta(u) \in \{b_1, b_2, \dots, b_{j-1}\}$ . If  $\beta(u) = b_l$  with  $b_j > b_l > b_i$ , then notice that there already exists a neighbour  $y$  of  $z$  with  $\alpha(y) = l$  and  $\beta(y) < \beta(z)$ , because the refined greedy algorithm set  $\alpha(z) = j$ . Note that  $P_z \cup zy$  is a decreasing path from  $v$  to  $y$ , which implies that  $y$  is a forced vertex. Clearly,  $u \neq y$  as  $\alpha(u) \neq \alpha(y)$ . Because of our induction hypothesis,  $\beta(y) = b_l$  and there is a colourful cycle containing the path  $P_z \cup zy$  as a subpath. As  $\beta(u) = \beta(y)$ ,  $u$  is outside this cycle but is a neighbour of  $z$ . This contradicts Lemma 16. Therefore,  $\beta(u) \leq b_i$ . Consider the decreasing path  $y_i y_{i-1} \dots y_1$  where  $y_i = u$ , and for  $s \in \{1, 2, \dots, i-1\}$ ,  $\alpha(y_s) = s$  and  $\beta(y_s) < \beta(y_{s+1})$  which exists by Lemma 3. Again by Lemma 3, there exists a decreasing path  $Q$  starting from  $v$  whose vertices other than  $v$  have exactly the labels in  $\{i+1, i+2, \dots, k\}$  that are not seen on  $P_z$ . By the induction hypothesis, we can now see that every colour in  $\{b_{i+1}, b_{i+2}, \dots, b_k\}$  occurs exactly once in the path  $Q \cup P_z$ . Since  $y_i y_{i-1} \dots y_1$  is a decreasing path in which every vertex has colour at most  $b_i$ , we can conclude that the path  $P' = Q \cup P_z \cup zy_i y_{i-1} \dots y_1$  is a colourful path. By Observation 7, the graph induced by  $V(P')$  is a colourful cycle containing  $v$ , which we shall call  $C'$ . By Corollary 20, we know that  $C'$  contains only primary colours. Now, if  $\beta(u) < b_i$ , then because  $u y_{i-1} \dots y_1$  was a decreasing path, it should mean that  $\beta(y_1) < b_1$ , which is a contradiction. Thus,  $\beta(u) = b_i$  and  $C'$  is a cycle containing  $P$  as a subpath and which contains only forced vertices and primary colours (note that each  $y_s$ , for  $1 \leq s \leq k-1$ , is a forced vertex as there is the decreasing path  $P_z \cup zy_i y_{i-1} \dots y_s$  from  $v$  to  $y_s$ ).  $\square$

**Lemma 22.** *If  $P_1$  and  $P_2$  are two decreasing paths ( $P_1 \neq P_2$ ) that start from a forced vertex  $u$  and meet at a vertex  $z$ , then*

- (i)  $\beta(z) = b_1$  and  $\alpha(z) = 1$
- (ii)  $u = v$ , and
- (iii)  $P_1 \cup P_2$  is a colourful cycle.

*Proof.* Since  $u$  is a forced vertex there exists a decreasing path  $P$  from  $v$  to  $u$ .

We shall first show that the paths  $P_1$  and  $P_2$  cannot see exactly the same set of colours. Suppose for the sake of contradiction that they do. As the paths  $P_1$  and  $P_2$  are different, there is a vertex in one of these paths that is not in the other. Let us assume without loss of generality that there is a vertex in the path  $P_2$  that is not present in  $P_1$ . We denote by  $x'$  the first vertex (when walking from  $u$ ) on  $P_2$  that is not present in  $P_1$ . Let  $x$  be the predecessor of  $x'$  on  $P_2$ . Clearly,  $x$  is also in  $P_1$ . Let  $x''$  denote the successor of  $x$  on  $P_1$ . As  $P_1$  and  $P_2$  are both decreasing paths with the same set of colours, it must be the case that  $\beta(x') = \beta(x'')$ . From Lemma 21, we know that there is a colourful cycle that contains  $P \cup P_1$  as a subpath. Clearly, this colourful cycle contains  $x''$  and as  $\beta(x') = \beta(x'')$ , this cycle does not contain  $x'$ . But now this colourful cycle and  $x'$  contradict Lemma 16.

Let  $\alpha(z) = i$ . By Lemma 21, we have  $\beta(z) = b_i$ . Let  $Y$  be the decreasing path  $y_i y_{i-1} \dots y_1$  (guaranteed to exist by Lemma 3) where  $y_i = z$ , and for  $s \in \{1, 2, \dots, i-1, i\}$ ,  $\alpha(y_s) = s$  and  $\beta(y_s) = b_s$  (by Lemma 21; note that each  $y_s$  is a forced vertex because of the decreasing path  $P \cup P_1 \cup y_i y_{i-1} \dots y_s$  from  $v$  to it). Again by Lemma 3, for  $j \in \{1, 2\}$ , there exists a decreasing path  $Q_j$  starting from  $v$  whose vertices other than  $v$  have exactly the labels in  $\{i+1, i+2, \dots, k\}$  that are not seen on  $P \cup P_j$ . Note that all the vertices in  $P, P_j, Q_j$  and  $Y$  are forced vertices and therefore by Lemma 21, they will all have primary colours. Furthermore, the colours on the path  $Q_j$  are exactly the primary colours greater than  $b_i$  that are not seen on the path  $(P \cup P_j) - v$  and the path  $Y$  will contain all the primary colours less than or equal to  $b_i$ . Let  $w_j$  denote the endpoint of  $Q_j$  that is not  $v$ . Since  $P \cup P_j \cup Y \cup Q_j$  is a colourful path, we have  $w_j y_1 \in E(G)$  for  $j \in \{1, 2\}$ . Observe that  $P_1$  and  $P_2$  are two paths such that in at least one of them, we have a vertex that is not present in the other. Since these paths also have the same endpoints, it is clear that there is a cycle in  $G$  using only edges in  $E(P_1) \cup E(P_2)$ . Note that as  $P_1$  and  $P_2$  do not see exactly the same set of colours, the paths  $Q_1 \cup w_1 y_1$  and  $Q_2 \cup w_2 y_1$  are different. Now, following the same reasoning as above, one can see that there is at least one cycle in  $G$  whose edges are from  $E(Q_1 \cup w_1 y_1) \cup E(Q_2 \cup w_2 y_1)$ . Since girth of  $G$  is  $k$ , we now have the following inequalities.

$$\begin{aligned} \|P_1\| + \|P_2\| &\geq k \\ \|Q_1\| + \|Q_2\| + 2 &\geq k \end{aligned}$$

As  $P \cup P_j \cup Y \cup Q_j \cup w_j y_1$  is a colourful cycle for  $j \in \{1, 2\}$ , we have the following equalities.

$$\begin{aligned} \|P\| + \|P_1\| + \|Q_1\| + \|Y\| + 1 &= k \\ \|P\| + \|P_2\| + \|Q_2\| + \|Y\| + 1 &= k \end{aligned}$$

Summing the first two inequalities, we get,

$$\|P_1\| + \|P_2\| + \|Q_1\| + \|Q_2\| + 2 \geq 2k$$

and by adding the third and fourth, we get,

$$2\|P\| + \|P_1\| + \|P_2\| + \|Q_1\| + \|Q_2\| + 2\|Y\| + 2 = 2k.$$

Combining the previous two equations, we have,

$$2\|P\| + 2\|Y\| \leq 0$$

which implies that  $\|Y\| = 0$  and  $\|P\| = 0$ . Therefore, we have  $\alpha(z) = 1$  (and hence,  $\beta(z) = b_1$ ) and also  $u = v$ . This proves (i) and (ii).

This means that  $Y$  contains just the vertex  $z$  and that  $P_1$  and  $P_2$  are both paths that start at  $v$  and end at  $z$ . Now, by our earlier observation,  $z$  is in a colourful cycle  $C_1 = P_1 \cup Q_1 \cup w_1 z$  and also in a colourful cycle  $C_2 = P_2 \cup Q_2 \cup w_2 z$ . There exists a vertex  $z' \in C_1$  with  $\beta(z') = b_2$  and a vertex  $z'' \in C_2$  such that  $\beta(z'') = b_2$ . Because each of  $C_1$  and  $C_2$  is the union of two decreasing paths from  $v$ , one can conclude that  $z'$  is a neighbour of  $z$  in  $C_1$  and  $z''$  is a neighbour of  $z$  in  $C_2$ . Applying Lemma 16 to either one of the cycles  $C_1$  or  $C_2$ , it can be seen that there can only be one neighbour of  $z$  coloured  $b_2$ , and therefore,  $z' = z''$ . Thus, both  $C_1$  and  $C_2$  contain the vertex  $z'$ . Let  $R_j$ , for  $j \in \{1, 2\}$ , be the subpath of  $C_j$  with  $v$  and  $z'$  as endvertices that does not contain  $z$ . Clearly,  $R_j$  is a decreasing path. Therefore,  $R_1$  and  $R_2$  are two decreasing paths that start at  $v$  and meet at  $z'$ . If  $R_1 \neq R_2$ , then since  $\beta(z') = b_2$ , we have a contradiction to (i). Therefore, it must be the case that  $R_1 = R_2$ , which implies that one of  $P_1 - z = P_2 - z$ ,  $Q_1 = Q_2$ ,  $P_1 - z = Q_2$  or  $P_2 - z = Q_1$  is true. Since  $P_1 \neq P_2$ , we know that  $P_1 - z \neq P_2 - z$  and because  $P_1$  and  $P_2$  do not contain the same set of colours, we also have  $Q_1 \neq Q_2$ . Therefore, we can conclude that either  $P_1 - z = Q_2$  or  $P_2 - z = Q_1$ . But in either case, we have that the colours on the path  $P_1 - \{v, z\}$  are exactly all the primary colours that are absent in  $P_2$ . This can only mean that  $P_1 \cup P_2$  is a colourful cycle. This proves (iii).  $\square$

**Corollary 23.** *The graph induced by all the forced vertices other than those coloured  $b_1$  is a tree.*

*Proof.* Suppose for the sake of contradiction that there is a cycle containing only forced vertices with colours other than  $b_1$ . Let  $u$  be the vertex in the cycle with the least colour in the colouring  $\beta$  and let  $u'$  and  $u''$  be its two neighbours on the cycle. Since  $u'$  and  $u''$  are both forced vertices there are decreasing paths  $P'$  and  $P''$  that start at  $v$  and end at  $u'$  and  $u''$  respectively. Let  $P_1$  be the decreasing path obtained by adding  $u$  to  $P'$  and  $P_2$  be the decreasing path obtained by adding  $u$  to  $P''$ . Since  $P' \neq P''$ , we clearly have  $P_1 \neq P_2$ . Thus,  $P_1$  and  $P_2$  are two decreasing paths starting at  $v$  and meeting at  $u$  and because  $\beta(u) \neq b_1$ , we have a contradiction to Lemma 22(i).  $\square$

Given any vertex  $w$ , we define  $\rho(w) = v_{k-1}$  if  $\beta(w) = b_k$  and  $\rho(w) = v_k$  otherwise.

Let  $X \subseteq \{b_1, b_2, \dots, b_k\}$  be any set of primary colours. Let  $L = \{i \mid b_i \in X\}$ . From Lemma 3, we know that there is a decreasing path starting from  $v_k$  consisting only of forced vertices and which sees exactly the labels in  $L \cup \{k\}$ . Applying Lemma 21 to the vertices of this path, we have that this path sees exactly the colours in  $X \cup \{b_k\}$ . Thus for any set of colours  $X \subseteq \{b_1, b_2, \dots, b_k\}$ , there exists a decreasing path starting from  $v_k$ , made up of forced vertices and which sees exactly the colours in  $X \cup \{b_k\}$ . Note that if  $b_k \notin X$ , we can apply the same argument to  $v_{k-1}$  and  $X$  and conclude that there is a decreasing path starting at  $v_{k-1}$  consisting only of forced vertices and which sees exactly the colours in  $X \cup \{b_{k-1}\}$ . Suppose that  $b_k \in X$ . Then let  $X' = X \setminus \{b_k\}$ . From our above observation, there is a decreasing path starting from  $v_k$ , consisting of only forced vertices and which sees exactly the colours in  $X'$ . This path, together with the edge  $v_{k-1}v_k$  gives us an almost decreasing path starting from  $v_{k-1}$  that consists only of forced vertices and which sees exactly the colours in  $X \cup \{b_{k-1}\}$ . Thus, for any set of primary colours  $X$ , we have almost decreasing paths starting from both  $v_k$  and  $v_{k-1}$  consisting of only forced vertices and which see exactly the colours in  $X \cup \{b_k\}$  and  $X \cup \{b_{k-1}\}$  respectively. This gives us the following observation.

**Observation 24.** *For any vertex  $w$ , and for any set of colours  $X \subseteq \{b_1, b_2, \dots, b_k\}$ , there exists an almost decreasing path starting from  $\rho(w)$ , made up of forced vertices and which sees exactly the colours in  $X \cup \{\beta(\rho(w))\}$ .*

Suppose that  $P$  is the decreasing path from  $v_k$  to a forced vertex  $y$ . Clearly, if  $v_{k-1}$  is on this path, then the subpath of  $P$  from  $v_{k-1}$  to  $y$  is a decreasing path from  $v_{k-1}$  to  $y$ . If  $v_{k-1}$  is not on  $P$ , then  $v_{k-1}v_k \cup P$  is an almost decreasing path starting from  $v_{k-1}$  and ending at  $y$ . As there are also decreasing paths from  $v_k$  to every other forced vertex, we now have the following observation.

**Observation 25.** *Given any vertex  $w$  and any forced vertex  $y$ , there exists an almost decreasing path starting from  $\rho(w)$  and ending at  $y$ .*

**Lemma 26.** *If  $w$  is an optional vertex such that  $N(w)$  contains a forced vertex  $u$ , then  $N(w) = \{u, y\}$  where  $\{\beta(u), \beta(y)\} = \{b_1, b_2\}$ . Moreover, there are decreasing paths  $P_u$  from  $\rho(w)$  to  $u$  and  $P_y$  from  $\rho(w)$  to  $y$  such that  $P_u \cup uw \cup P_y$  is a colourful cycle.*

*Proof.* We shall first show that  $\beta(u) \in \{b_1, b_2\}$ . Suppose that  $\beta(u) \notin \{b_1, b_2\}$ .

We claim that there is a forced vertex  $z \in N(w)$  with  $\beta(z) \notin \{b_1, b_2\}$  such that there is an almost decreasing path  $P_z$  from  $\rho(w)$  to  $z$  that does not contain  $\beta(w)$ . Let  $P_u$  be the almost decreasing path from  $\rho(w)$  to  $u$  that exists by Observation 25. If  $\beta(w)$  is not in  $P_u$ , then we can set  $z = u$  and we are done. Suppose  $\beta(w) \in P_u$ . By Observation 24, there exists a path  $P'_u$  that starts from  $\rho(w)$  and whose vertices other than  $\rho(w)$  have exactly the colours that are missing in  $P_u$ . Let  $u'$  be the endvertex of  $P'_u$  other than  $\rho(w)$ . Clearly,  $P_u \cup P'_u$  is a colourful path and therefore, by Observation 7, we have  $uu' \in E(G)$ . By Observation 24, there exists an almost decreasing path  $Q$  consisting of forced vertices starting from  $\rho(w)$  that sees exactly the colours in  $P_u$  other than  $\{\beta(w), \beta(u)\}$ . Now,  $\{wu, uu'\} \cup P'_u \cup Q$  is a colourful path and therefore the endvertex of  $Q$  is a forced vertex adjacent to  $w$ . We claim that this endvertex of  $Q$  can be chosen as  $z$ . Clearly,  $Q$  is an almost decreasing path from  $\rho(w)$  to  $z$  that does not contain  $\beta(w)$  and this can be considered to be the required path  $P_z$ . Also, since  $\beta(u) \notin \{b_1, b_2\}$ , no vertex in  $P_u$  has colour  $b_1$  or  $b_2$ , implying that no vertex in the almost decreasing path  $Q$  has either of these colours. Therefore,  $\beta(z) \notin \{b_1, b_2\}$ . Hence  $z$  is a neighbour of  $w$  of the required type.

Let  $z$  be a forced vertex in  $N(w)$  such that there is an almost decreasing path  $P_z$  from  $\rho(w)$  to  $z$  that does not contain  $b_1, b_2$  or  $\beta(w)$ . By Observation 24, there exists an almost decreasing path  $Q$  starting from  $\rho(w)$  whose vertices other than  $\rho(w)$  have exactly the primary colours that are not in the path  $P_z \cup zw$ . Note that  $b_1$  and  $b_2$  will be present in  $Q$ . Let  $u_3, u_2, u_1$  be last three vertices of  $Q$  in that order (i.e.,  $u_1$  is the endvertex of  $Q$  other than  $\rho(w)$ ). Since  $Q$  is



an almost decreasing path, we have  $\beta(u_1) = b_1$  and  $\beta(u_2) = b_2$ . Since  $wz \cup P_z \cup Q$  is a colourful path, we have by Observation 7 that  $wu_1 \in E(G)$ . By Observation 24, there exists an almost decreasing path  $Q'$  that starts at  $\rho(w)$  and whose vertices other than  $\rho(w)$  see exactly the primary colours that are not present in  $Q$ . Note that the paths  $Q'$  and  $P_z \cup zw$  see exactly the same colours. Let the endvertex of  $Q'$  other than  $\rho(w)$  be  $w'$ . As  $Q' \cup Q$  is a colourful path, we have  $w'u_1 \in E(G)$ . By applying Lemma 3 and Lemma 21 together, we know that there is a vertex  $p$  with  $\beta(p) = b_1$  that is adjacent to  $u_3$ . Let  $R$  be the union of the subpath of  $Q$  from  $\rho(w)$  to  $u_3$  and the edge  $u_3p$ . By Observation 24, there exists an almost decreasing path  $R'$  starting from  $\rho(w)$  and whose vertices other than  $\rho(w)$  have exactly the primary colours not present in  $R$ . Let  $r$  be the end vertex of  $R'$ . Clearly,  $\beta(r) = b_2$ , and because  $R \cup R'$  is a colourful path, we have  $pr \in E(G)$ . Now, observe that the path  $zw \cup P_z \cup R \cup pr$  is also a colourful path, and therefore, we have  $wr \in E(G)$ . Similarly, the path  $Q' \cup R \cup pr$  is also a colourful path, leading to the conclusion that  $w'r \in E(G)$ . Recall that  $w \neq w'$  as one is a forced vertex while the other is an optional vertex and  $u_1 \neq r$  as  $\beta(u_1) \neq \beta(r)$ . We now have a four cycle  $wrw'u_1w$  in  $G$ , which is a contradiction.

Therefore, we can conclude that  $\beta(u) \in \{b_1, b_2\}$ . By Observation 25, there exists an almost decreasing path  $R_u$  from  $\rho(w)$  to  $u$ . By Observation 24, there exists an almost decreasing path  $R'_u$  that starts from  $\rho(w)$ , ends at a vertex  $u'$ , and whose vertices other than  $\rho(w)$  have exactly the primary colours that are not in  $R_u$ . By Observation 7, we know that  $uu' \in E(G)$  and that  $G[V(R_u) \cup V(R'_u)]$  is a colourful cycle. Therefore, there exists a path  $P \in \{R_u, R'_u \cup u'u\}$  such that  $P$  does not contain  $\beta(w)$ . By Observation 24, there exists an almost decreasing path  $Q$  consisting of forced vertices that starts from  $\rho(w)$ , ends at a vertex  $y$ , and whose vertices other than  $\rho(w)$  see exactly the primary colours that are not in  $V(P) \cup \{w\}$ . Now, notice that  $wu \cup P \cup Q$  is a colourful path and therefore  $y$  will be adjacent to  $w$ . This tells us that  $\beta(y) \in \{b_1, b_2\}$ . If  $P = R_u$ , then it is clear that  $P$  is an almost decreasing path from  $\rho(w)$  to  $u$ . If, on the other hand,  $P = R'_u \cup u'u$ , then notice that since  $wu \cup P \cup Q$  is a colourful path which contains the vertices  $u, y$  with  $\{\beta(u), \beta(y)\} = \{b_1, b_2\}$ , we have  $\beta(u') > \beta(u)$ . Since  $R'_u$  is an almost decreasing path, this tells us that  $P = R'_u \cup u'u$  is also an almost decreasing path. Also, since  $wu \cup P \cup Q$  is a colourful path, both the almost decreasing paths  $P$  and  $Q$  do not see the colour  $\beta(w)$ , and hence they are both decreasing paths. Thus we can set  $P_u = P$  and  $P_y = Q$  as the required decreasing paths. Now, the application of Lemma 16 to the colourful cycle  $wu \cup P \cup Q \cup yw$  and vertex  $w$  tells us that these are the only two vertices with colours  $b_1$  and  $b_2$  in  $N(w)$ . Thus, we can conclude that  $N(w) = \{u, y\}$ .  $\square$

**Lemma 27.** *Every optional vertex is adjacent to a forced vertex.*

*Proof.* Suppose that there are optional vertices in  $G$  that are not adjacent to any forced vertex. Then let  $w$  be the optional vertex among them that is closest to a forced vertex. Since  $G$  is connected, we can conclude that  $w$  is adjacent to some optional vertex  $w'$  that has a forced vertex in  $N(w')$ . By Lemma 26, we know that there is a colourful cycle  $C$  containing  $w'$  in which every vertex other than  $w'$  is a forced vertex. Let  $z$  be that vertex in  $C$  with the highest colour in  $\{b_1, b_2, \dots, b_k\} \setminus \{\beta(w), \beta(w')\}$ . Let  $P$  be that subpath of  $C$  between  $w'$  and  $z$  that does not contain  $\beta(w)$ . By Lemma 3 and Lemma 21, there exists a decreasing path  $P'$  starting from  $z$  whose vertices other than  $z$  have exactly the primary colours that are not present in the path  $P \cup w'w$ . Clearly,  $ww' \cup P \cup P'$  is a colourful path and therefore the endvertex  $y$  of  $P'$  is adjacent to  $w$ . Notice that  $P'$  was a path consisting entirely of forced vertices and therefore  $y$  is a forced vertex in the neighbourhood of  $w$ , which is a contradiction.  $\square$

**Lemma 28.** *The set of optional vertices form an independent set.*

*Proof.* We shall first observe that like forced vertices, every optional vertex also has a primary colour. Consider an optional vertex  $w$ . By Lemma 27, we know that there is a forced vertex  $y$  in  $N(w)$ . Notice that by Lemma 21, the forced vertex  $y$  is in a colourful cycle containing only forced vertices and primary colours. Therefore, the optional vertex  $w$  is adjacent to a colourful cycle containing only primary colours. Thus, by Corollary 17, we can conclude that the  $\beta(w)$  is a primary colour.

We shall now prove the statement of the lemma. Let  $w_1$  and  $w_2$  be optional vertices such that  $w_1w_2 \in E(G)$ . Let us assume without loss of generality that  $\beta(w_1) < \beta(w_2)$ . Suppose first that  $\beta(w_1) = b_{k-1}$  and  $\beta(w_2) = b_k$ . In this case,  $\rho(w_1) = v_k$  and  $\rho(w_2) = v_{k-1}$ . Let  $y, z$  be forced vertices in  $N(w_1)$  and  $y', z'$  be forced vertices in  $N(w_2)$  such that  $\beta(y) = \beta(y') = b_1$  and  $\beta(z) = \beta(z') = b_2$ . Note that  $y \neq y'$  and  $z \neq z'$  as  $G$  is triangle-free. By Lemma 26, we know that  $y, z, y', z'$  exist and also that there is a colourful cycle  $C$  containing vertices  $w_1, v_k$  and the edges  $w_1y$  and  $w_1z$ . Similarly, there is a colourful cycle  $C'$  containing vertices  $w_2, v_{k-1}$  and the edges  $w_2y'$  and  $w_2z'$ . Let  $P, Q$  be the subpaths of  $C$  with endvertices  $v_k$  and  $w_1$  that contain the edges  $w_1y, w_1z$  respectively. Similarly, let  $P', Q'$  be the subpaths of  $C'$  with endvertices  $v_{k-1}$  and  $w_2$  that contain the edges  $w_2y', w_2z'$  respectively. Let  $u$  be the vertex with

colour  $b_{k-2}$  on the cycle  $C$ . Since from Lemma 26, it is clear that  $P - w_1$  and  $Q - w_1$  are both decreasing paths, it has to be the case that  $u$  is a neighbour of  $v_k$  on either  $P$  or  $Q$ . Suppose that  $u$  is on  $P$ . By Lemma 3, we know that there is a decreasing path  $R$  starting from  $u$  such that its vertices other than  $u$  see exactly the colours that are absent on the path  $P$ . Notice that  $w_2 w_1 \cup (P - v_k) \cup R$  is a colourful path and hence the endvertex of  $R$  is adjacent to  $w_2$ . Clearly, the endvertex of  $R$  has colour  $b_2$  and therefore by Lemma 26, this vertex is none other than  $z'$ . Now, the paths  $v_k u \cup R$  and  $v_k v_{k-1} \cup (Q' - w_2)$  are both distinct decreasing paths (note that  $v_{k-1}$  is present in one of the paths but not in the other) that meet at the vertex  $z'$ . Since  $\beta(z') = b_2$ , we now have a contradiction to Lemma 22(i). Therefore, we can conclude that  $u$  is on  $Q$ . Again by Lemma 3, we know that there is a decreasing path  $R$  starting from  $u$  whose vertices other than  $u$  see exactly the colours that are absent on the path  $Q$ . Notice that  $w_2 w_1 \cup (Q - v_k) \cup R$  is a colourful path and hence the endvertex of  $R$  is adjacent to  $w_2$ . As it is clear that this endvertex of  $R$  has colour  $b_1$ , by Lemma 26, this vertex is none other than  $y'$ . Now, the paths  $v_k u \cup R$  and  $v_k v_{k-1} \cup (P' - w_2)$  are both distinct decreasing paths that meet at the vertex  $y'$ . By Lemma 22(iii), we know that these paths together constitute a colourful cycle. But the colour  $b_2$  is not present in either of these paths, which is a contradiction.

By the above arguments, we can assume that  $\{\beta(w_1), \beta(w_2)\} \neq \{b_{k-1}, b_k\}$ . Therefore,  $\beta(w_1) < b_{k-1}$  which means that  $\rho(w_1) = v_k$ . By Lemma 26, we know that there is a colourful cycle  $C$  containing  $w_2$  and  $\rho(w_2)$ . Let  $P$  be that subpath of  $C$  between  $\rho(w_2)$  and  $w_2$  that does not contain  $\beta(w_1)$  (note that  $\beta(\rho(w_2)) \neq \beta(w_1)$ , implying that such a path exists) and let  $Q$  be that subpath that does. Also, we shall denote by  $y$  the neighbour of  $w_2$  on  $P$  and by  $z$  the neighbour of  $w_2$  on  $Q$ . By Observation 24, there exists an almost decreasing path  $P'$  starting from  $\rho(w_2)$  in which the vertices other than  $\rho(w_2)$  see exactly the colours that are not present in the path  $P \cup w_2 w_1$ . Clearly,  $w_1 w_2 \cup P \cup P'$  is a colourful path and therefore the endvertex  $y'$  of  $P'$  is adjacent to  $w_1$ . Note that  $y'$  is different from  $y$  and  $z$  as otherwise, there would be a triangle in the graph. Let  $Q'$  be the almost decreasing path starting from  $\rho(w_2)$ , that exists by Observation 24, such that its vertices other than  $\rho(w_2)$  see exactly the missing colours from  $P'$  other than  $\beta(w_1)$ . Notice that  $Q'$  contains exactly the colours from  $P$ . Let  $z'$  be the endvertex of  $Q'$  other than  $\rho(w_2)$ . Clearly,  $z' \neq w_2$  as one is a forced vertex while the other is an optional vertex. Observe that  $w_1 y' \cup P' \cup Q'$  is a colourful path and hence  $z' w_1 \in E(G)$ . As before,  $z'$  is distinct from  $y$  and  $z$  as otherwise, there would be a triangle in the graph.

Let  $Q_z$  be the subpath of  $Q$  from  $z$  to  $\rho(w_2)$ . Recall that  $Q'$  contains exactly the colours from  $P$ . As the path  $P \cup Q_z$  was a colourful path, we can infer that  $Q' \cup Q_z$  is also a colourful path. Hence there is an edge  $zz' \in E(G)$ . But since  $z \in N(w_2)$  and  $z' \in N(w_1)$ , we have the 4-cycle  $w_2 w_1 z' z w_2$  in  $G$ , which is a contradiction. Therefore we conclude that for any two optional vertices  $w_1$  and  $w_2$ , we have  $w_1 w_2 \notin E(G)$ . We have thus proved that the set of optional vertices is an independent set.  $\square$

Let  $w$  be an optional vertex. From Lemma 27 and Lemma 26, we know that  $N(w) = \{y, z\}$  where  $\beta(y) = b_1$ ,  $\beta(z) = b_2$  and both  $y$  and  $z$  are forced vertices. This also tells us that  $\beta(w) \notin \{b_1, b_2\}$ . Let  $T$  denote the graph induced in  $G$  by the forced vertices other than those coloured  $b_1$ . From Corollary 23, we know that  $T$  is a tree. By what we have observed above, it is clear that each optional vertex has exactly one neighbour in  $V(T)$ . Also, Lemma 28 tells us that the optional vertices form an independent set. Therefore,  $V(T)$  and the set of optional vertices together induce a tree in  $G$ . Since optional vertices do not have colour  $b_1$  as observed above, we can conclude that the subgraph induced in  $G$  by the vertices other than those coloured  $b_1$  is a tree. This implies that  $\chi(G) \leq 3$ , which is a contradiction to the fact that  $\chi(G) = k > 4$ . This completes the proof of Theorem 6 for the case when  $k > 4$ .

#### 4. Conclusion

We have shown in this paper that for any properly coloured graph  $G$  with  $g(G) \geq \chi(G)$ , there exists an induced colourful path on  $\chi(G)$  vertices in  $G$ . The question of whether every properly coloured graph  $G$  contains an induced colourful path on  $\chi(G)$  vertices remains open for the case  $3 < g(G) < \chi(G)$ .

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